IMEX Runge-Kutta schemes for delay differential equations

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Numerical methods for problems with functional dependence

Consider a pair of two Runge-Kutta methods defined by the arrays

$$\begin{array}{c|cccccc} 0 & 0 & 0 & & \cdots & 0 \\ c_2 & a_{21} & a_{22} & 0 & \cdots & 0 \\ c_3 & a_{31} & a_{32} & a_{33} & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s \end{array}, \qquad \begin{array}{c|cccccc} 0 & 0 & 0 & & \cdots & 0 \\ c_2 & \widehat{a}_{21} & 0 & & \cdots & 0 \\ c_3 & \widehat{a}_{31} & \widehat{a}_{32} & 0 & & \vdots \\ \vdots & \vdots & & & \ddots & 0 \\ c_s & \widehat{a}_{s1} & \widehat{a}_{s2} & \cdots & \widehat{a}_{s,s-1} & 0 \\ \hline & \widehat{b}_1 & \widehat{b}_2 & \cdots & \widehat{b}_{s-1} & \widehat{b}_s \end{array},$$

where a_{ij} and $\widehat{a}_{ij}$ are assumed to satisfy

$$c_i = \sum_{j=1}^{i} a_{ij} = \sum_{j=1}^{i-1} \widehat{a}_{ij}.$$

An IMEX (implicit-explicit) Runge-Kutta method for the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu(t) + g\Big(t, u(t)\Big)$$

is given by

$$\begin{aligned} U_{n,i} &= u_n + \Delta t \sum_{j=1}^{i} a_{ij}\, LU_{n,j} + \Delta t \sum_{j=1}^{i-1} \widehat{a}_{ij}\, g\Big(t_n + c_j \Delta t, U_{n,j}\Big), \\ u_{n+1} &= u_n + \Delta t \sum_{i=1}^{s} b_i\, LU_{n,i} + \Delta t \sum_{i=1}^{s} \widehat{b}_i\, g\Big(t_n + c_i \Delta t, U_{n,i}\Big). \end{aligned}$$

We first discuss the application of the method to delay differential equations (DDEs) of the form

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu(t) + g\Big(t, u(t), u(t-\tau)\Big),$$

where $\tau > 0$ is a constant delay. In particular, we study stability of the method using a linear test equation. We also consider a continuous extension of the method and discuss the application of the scheme to more general DDEs.